Lagrangian isotopies in Stein manifolds

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1 Introduction

Studying the space of Lagrangian submanifolds is a fundamental problem in symplectic topology. Lagrangian spheres appear naturally in the Leftschetz pencil picture of symplectic manifolds.

In this paper we demonstrate the uniqueness up to Hamiltonian isotopy of the Lagrangian spheres in some 4-dimensional Stein symplectic manifolds. The most important example is the cotangent bundle of the 2-sphere, T^*S^2, with its standard symplectic structure.

We recall that if a convex symplectic manifold has a boundary of contact-type, then we can perform surgery operations on the manifold by adding handles to the boundary. In the 4-dimensional case these handles can be of index 1 or 2. Our other examples are symplectic manifolds formed by adding 1-handles to a unit cotangent bundle T^1S^2. Questions regarding Lagrangian isotopy classes are independent of which metric we use to define a unit tangent bundle or of any choices involved in adding 1-handles.

Theorem 1 *Let M be T^*S^2 or the result of adding any number of 1-handles to T^1S^2 and $L \subset M$ be a Lagrangian sphere. Then there exists a Hamiltonian*

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diffeomorphism of M mapping L onto the zero-section.

The statement is false even if we add to $T^1 S^2$ a single 2-handle along the Legendrian curve in a single fiber of the boundary. The result of this surgery operation is a plumbing M of two copies of $T^1 S^2$. The symplectic manifold M has two Lagrangian spheres L_1 and L_2 coming from the zero-sections in the $T^1 S^2$. Now, associated to any Lagrangian sphere L is a compactly supported symplectomorphism τ_L (a generalized Dehn Twist) whose square is smoothly but not necessarily symplectically isotopic to the identity. Thus $\tau_{L_2}^2(L_1)$ is a Lagrangian sphere in M which is smoothly isotopic to L_1. However, as demonstrated by P. Seidel in [8], a Floer homology computation shows that $\tau_{L_2}^2(L_1)$ is not Hamiltonian isotopic to L_1.

We will establish the theorem by utilizing an existence result for almost-complex structures on $S^2 \times S^2$ with convenient properties, taken from [4], and a fact about diffeomorphisms of the 2-sphere.

In section 2 we prove our result on the diffeomorphisms of S^2. In section 3, by using the conclusions of [4], we reduce our theorem in the case of $M = T^* S^2$ to the statements in section 2. In section 4 we deal with the addition of handles. This involves slightly generalizing the results from [4] so we will review them again there.

As yet we are unable to prove any similar results for higher genus Lagrangian surfaces, but we can make some remarks about the case of $\mathbb{R}P^2$. The results of [4] show that any Lagrangian sphere L in $S^2 \times S^2$ homotopic to the antidiagonal $\overline{\Delta}$ is in fact Lagrangian isotopic to $\overline{\Delta}$. In this paper we will show that if L is disjoint from the diagonal Δ then the Lagrangian isotopy can be chosen to lie in $S^2 \times S^2 \setminus \Delta$. Now, the involution σ of $S^2 \times S^2$ interchanging the two factors has fixed-point set equal to Δ and restricts to the antipodal map on $\overline{\Delta}$. If L is invariant under σ then the isotopy can also be chosen to be σ-equivariant. Now, quotienting out by σ, we observe that $S^2 \times S^2 \setminus \Delta$ is a double-cover of a unit cotangent bundle of $\mathbb{R}P^2$ and Lagrangian spheres in $T^* \mathbb{R}P^2$ homotopic

to the zero-section therefore correspond to σ-invariant Lagrangian spheres in $S^2 \times S^2 \setminus \Delta$ homotopic to $\overline{\Delta}$. Hence we have the following corollary.

Corollary 2 *A Lagrangian $\mathbb{R}P^2$ homotopic to the zero-section in $T^*\mathbb{R}P^2$ must be Hamiltonian isotopic to the zero-section.*

A natural compactification of the (unit) cotangent bundle of $\mathbb{R}P^2$ is $\mathbb{C}P^2$. The uniqueness up to Hamiltonian isotopy of Lagrangian $\mathbb{R}P^2$s inside $\mathbb{C}P^2$ can be established by other methods. For example, the surgery technique described in [10] replaces a Lagrangian $\mathbb{R}P^2$ by a symplectic sphere, transforming $\mathbb{C}P^2$ into $S^2 \times S^2$. But the symplectic spheres in $S^2 \times S^2$ have been classified up to Hamiltonian isotopy in [9].

2   Diffeomorphisms of the two-sphere

In this section we let f denote a diffeomorphism of the 2-sphere S^2 and for a point $x \in S^2$ we denote its antipodal point by $-x$.

We say that a diffeomorphism f has the property $(*)$ if $f(x) \neq -x$ for all $x \in S^2$.

The aim of the section is to prove the following theorem.

Theorem 3 *Suppose that f has the property $(*)$. Then there exists an isotopy f_t, $0 \leq t \leq 1$, with $f_0 = \mathrm{id}$ and $f_1 = f$ such that f_t has property $(*)$ for all t.*

Proof of theorem

Let E denote an equator on S^2. The complement of E consists of two open disks H_1 and H_2 with $-H_1 = H_2$.

We observe that any diffeomorphism g with property $(*)$ and which preserves E is indeed isotopic to the identity through diffeomorphisms G_t also satisfying $(*)$. To construct such an isotopy, we first isotope g to the identity in a neighbourhood of E. Now the resulting map restricts to a compactly supported

diffeomorphism of H_1 and H_2. But compactly supported diffeomorphisms of the disk are isotopic to the identity (see for instance [11], page 205). Combining these isotopies we get the required isotopy of g. It satisfies $(*)$ since $-H_1 = H_2$.

Hence it suffices to find a suitable isotopy from f to a diffeomorphism preserving an equator E.

Now, the diffeomorphism is necessarily orientation preserving and so has a fixed point p_0. After a small perturbation we may assume that $f = \mathrm{id}$ near p_0. We first assume that f also satisfies $f(-p_0) = -p_0$, and so by another small perturbation can suppose that $f(x) = x$ also near $-p_0$. Then we can identify $S^2 \setminus \{p_0, -p_0\}$ with $\mathbb{R} \times S^1$ such that in these coordinates the antipodal map is given by

$$-(x, e^{i\theta}) = (-x, -e^{i\theta})$$

and f is equal to the identity on the complement of a compact set C. In these coordinates let the equator $E = \{x = 0\}$.

We construct our isotopy by applying the following lemma.

Lemma 4 *Let L_s, $-\infty < s < \infty$ be a 1-dimensional foliation of $\mathbb{R} \times S^1$ coinciding with $\{x = s\}$ outside C. For any N there exists a compactly supported isotopy f_t satisfying $(*)$ such that $f_0 = f$ and $f_t(z) \in f(L_{s+tN})$ for all $z \in L_s \cap C$, $0 \le t \le 1$.*

Given this, we can conclude as follows. First we apply the lemma with $L_s = \{x = s\}$ to find an isotopy to a new diffeomorphism, still denoted by f, such that $f(z) \in L_K$ for all $z \in E$, for K large. Let $N = f^{-1}(E)$. Then N is disjoint from E and so we can form another foliation L'_s which includes the circles N and E. After another application of the lemma, perhaps now with a larger value of K, we can isotope f to another diffeomorphism now satisfying $f(z) \in E$ for all $z \in E$.

Proof of lemma

As the condition on our isotopy is an open one, we may assume any necessary genericity properties for the diffeomorphism f with respect to the foliation L_s.

Suppose that $N > 0$. We can set $f_t(x, e^{i\theta}) = (x + Nt, e^{i\theta})$ for x large and positive. In general, for $r \in \mathbb{R}$, let a_r be a diffeomorphism of $\mathbb{R} \times S^1$ such that $a_r(L_s) = L_{s+r}$. Outside of C we may assume that $a_r(x, e^{i\theta}) = (x+r, e^{i\theta})$. Then we will define f_t by

$$f_t(z) = h_t f a_{Nt}(z)$$

where h_t is a diffeomorphism of $\mathbb{R} \times S^1$ which preserves the foliation $\{L_s\}$. We set $h_{t,s} = h_t|_{f(L_{s+Nt})}$.

Then we need to find smoothly varying $h_{t,s}$ such that $h_{t,s}(f(a_{tN}(z)) \neq -z$ for all $z \in L_s$, s and $0 \leq t \leq 1$.

For s very large we have $h_{t,s} = \mathrm{id}$ and it is required to show that we can extend these diffeomorphisms for all parameters s. Again since property $(*)$ is an open condition, we observe that once we have defined the h_{t,s_0} for some s_0 we can smoothly extend the functions to define $h_{t,s}$ for s slightly less than s_0.

Another observation is that an isotopy defined with the required properties for $s \geq s_1$, some s_1, can always be extended to an isotopy of $\mathbb{R} \times S^1$ satisfying the property $(*)$ and preserving the foliation by levels $f(L_s)$. To do this, we simply smoothly cut-off the vector field generating f_t in such a way that the cut-off still preserves the levels. Property $(*)$ still holds provided that the vector field is zero on $f(L_s)$ for $s < s_1 - \epsilon$, ϵ sufficiently small. The key point is that since the orbit of a point x on L_{s_1} avoids the point $-x$, it also avoids $-y$ for all y close to x.

For any s, as t increases from 0 to 1 there is a varying collection of points $I_{t,s} = f(L_{s+tN}) \cap -L_s$. The diffeomorphisms $h_{t,s}$ can be extended arbitrarily once they are defined on these intersections. We notice that $I_{t,s}$ is empty for s sufficiently large or small.

Now, for typical s' the families of points $I_{t,s}$, $0 \leq t \leq 1$, will vary continuously with s for s close to s'. For a fixed value of s they will consist of

a continuously varying set of points which at certain times appear or vanish in pairs. Continuous variation with s means that we have smooth families of diffeomorphisms

$$\phi_s : [0,1] \to [0,1]$$

$$g_{t,s} : f(L_{s'+tN}) \to f(L_{s+\phi_s(t)N})$$

$$b_{t,s} : L_{s'} \to L_s$$

such that $\phi_{s'} = \mathrm{id}$, $g_{t,s'} = \mathrm{id}$, $b_{t,s'} = \mathrm{id}$ and $g_{0,s}(f(z)) = f(b_{0,s}(z))$. They can be chosen such that $g_{t,s}(I_{t,s'}) = I_{\phi_s(t),s}$ and if $-z \in I_{t,s'}$, then $g_{t,s}(-z) = -b_{t,s}(z)$. The existence of such diffeomorphisms implies that if we have defined suitable $h_{t,s}$ for an s close to s' then we may define the $h_{t,s'}$ by

$$h_{t,s'}(f(a_{tN}(z))) = g_{t,s}^{-1}(h_{\phi_s(t),s}(f(a_{\phi_s(t)N}b_{t,s}(z)))).$$

At a finite collection of parameters s_i the pattern of intersections I_{t,s_i} will change from nearby values. Assuming f to be generic, the change will occur only near a single point in C for a single t parameter.

Suppose that s' is such a critical parameter. In the first case we consider the situation when $f(L_{s'})$ and $f(L_{s'+N})$ are transverse to $-L_{s'}$. Then since s' is critical there exists a σ with $s' < \sigma < s' + N$ with $f(L_\sigma)$ tangent to order three with $-L_{s'}$. For, if these tangencies are all of order at most two, then the intersections $I_{t,s}$ will indeed vary continuously with s for s close to s'. Genericity allows us to assume that such a tangency has order no more than three.

Let the tangency occur at a point p. We can choose coordinates (x,y) in $\mathbb{R}^2$ with $p = (0,0)$ such that the foliation $-L_s$ is given by $\{x = s' - s\}$ and the foliation $\{f(L_s)\}$ by $\{x = y^3 - sy \pm s\}$, where here we take the parameter $\sigma = 0$.

If the second foliation is $x = y^3 - sy + s$ then $f(L_s) \cap -L_{s_1}$ is a single point for all s close to 0 and $s_1 \geq s'$ but $f(L_s) \cap -L_{s_2}$ consists of three points for some $s > 0$ and $s_2 < s'$. In this case we can define the diffeomorphisms $g_{t,s}$ and $b_{t,s}$ as before for $s > s'$ and thus define $h_{t,s'}$. Recall that we are implicitly assuming that the $h_{t,s}$ can de defined for $s > s'$ but not necessarily for $s \leq s'$.

However, if the second foliation is given by $y = x^3 - sx - s$, then $f(L_s) \cap -L_{s_1}$ consists of three points for some $s > 0$ and $s_1 > s'$ but only one point for all s close to 0 and $s_1 \le s'$. Thus the diffeomorphisms $g_{t,s}$ do not exist as before for $s > s'$. Nevertheless we can define diffeomorphisms $g_{t,s}$ satisfying the same conditions as before away from a neighbourhood of $t = \sigma - s'$ and the point p. Near p and $t = \sigma - s'$ we extend the diffeomorphisms sending $I_{t,s'}$ to the first point in $I_{\phi_s(t),s}$, coherently ordering the intersection points along $-L_s$. This works as before to extend the $h_{t,s}$.

Finally suppose that $-L_{s'}$ is tangent to $f(L_{s'})$ or $f(L_{s'+N})$. We may now assume that this tangency is of second order and the diffeomorphisms $g_{t,s}$ and $b_{t,s}$ will exist as before for t away from 0 and 1. This is already enough in the case of a tangency with $f(L_{s'})$ since we can set $h_{t,s'} = \text{id}$ for t close to 0.

If $f(L_{s'+N})$ has a second order tancency with $-L_{s'}$ at a point p then $f(L_{s+N}) \cap -L_s$ is either empty or consists of two points for $s > s'$. In the first case it is easy to extend the $h_{t,s}$ to $s = s'$ simply ensuring that $f_1^{-1}(-p) \ne p$.

In the second case, suppose that $f_1(L_{s_1}) \cap -L_{s_1} = \{f_1(y), f_1(z)\}$ for some $s_1 > s'$. We can extend the isotopy to $L_{s'}$ if and only if the points $f(y), -y, -z, f(z)$ do not occur in that order along $-L_{s_1}$, this is the obstruction to the intersection points moving together. But recall that the isotopy does indeed extend with property $(*)$ to all L_s if we neglect the condition that $f_1(L_s) = f(L_{s+N})$, but still can require that $f_1(L_s) = f_1(L_r)$ for some $r > s$. The extended isotopy then still has the property that some $f_1(L_s)$ is tangent to $-L_s$ near the point p. Thus the obstruction must vanish and we can define $f_t(L_{s'})$ as required. This completes the proof of the lemma.

To complete the proof of Theorem 3 we must justify the assumption that $f(-p_0) = -p_0$. Specifically, assuming that $f = \text{id}$ in a neighborhood U of p_0 we need to find an isotopy of f to a diffeomorphism which is also equal to the identity near $-p_0$.

Now let F be the equator passing through p_0, $-p_0$ and $f(-p_0) \ne -p_0$, and

G be a small circle in U touching F only in a small interval about the point p_0. We can extend the circles G and F to a 1-parameter family of circles C_s, $-\infty < s < \infty$, with $F = C_0$, $G = C_1$ and $C_s \subset U$ whenever $|s| > 1$. Each of the circles will intersect F in a small interval around p_0 and away from p_0 they give a foliation of S^2. Then by a variation of Lemma 4 we may find an isotopy of f to another diffeomorphism f satisfying $f(F) = G$ with $f = \mathrm{id}$ still on a small neighborhood of p_0. Consider a path γ from $f(-p_0)$ to $-p_0$ following part γ_1 of G and then part γ_2 of the equator F avoiding p_0. In fact we may suppose that $\gamma_1 = f(\gamma_2)$. We observe that on the path γ we will never encounter points $f(z)$ then $-z$ in that order. This is clear because $\gamma_1 = f(\gamma_2)$ and γ_2 is disjoint from $-\gamma_2$. Hence, by following γ, we can find an isotopy g_t, supported in a small neighbourhood of γ and such that $g_0 = \mathrm{id}$ and $g_1(f(-p_0)) = -p_0$. Furthermore $g_t \circ f$ satisfies property $(*)$ and so gives our isotopy as required.

3   Lagrangian spheres in T^*S^2

Let L be a Lagrangian sphere in T^*S^2. This has self-intersection number -2 and so must be homotopic to the zero-section. By scaling in the fibers we may assume that $L \subset T^1S^2$. We will identify T^1S^2 with the complement of the diagonal Δ in $S^2 \times S^2$ with its standard split symplectic form $\omega = \omega_0 \oplus \omega_0$. Under this identification, the zero-section in T^1S^2 becomes the antidiagonal $\overline{\Delta}$. Thus our theorem in this case reduces to the following.

Theorem 5 *Given a Lagrangian sphere $L \subset S^2 \times S^2 \setminus \Delta$ homotopic to $\overline{\Delta}$, there exists a Hamiltonian isotopy of $S^2 \times S^2$ which fixes Δ and maps L onto $\overline{\Delta}$.*

Given an almost-complex structure J on $S^2 \times S^2$ tamed by ω, Gromov showed in [3] that there exist unique foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ by J-holomorphic curves in the classes $[S^2 \times \mathrm{pt}]$ and $[\mathrm{pt} \times S^2]$. With respect to the standard almost-complex structure $J_0 = i \oplus i$, these foliations are exactly $S^2 \times \mathrm{pt}$ and $\mathrm{pt} \times S^2$. The key lemma which we need from [4] is the following.

Lemma 6 *There exists a tame almost-complex structure J on $S^2 \times S^2$ such that each curve in the corresponding foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ intersects L transversally in a single point. The almost-complex structure J can be taken to agree with J_0 near Δ.*

The second statement was not included in [4] but is clearly true from the proof.

There exists a family of tame almost-complex structures J_t, $0 \leq t \leq 1$ on $S^2 \times S^2$ with $J_1 = J$ and, for all t, $J_t = J_0 = i \oplus i$ near Δ. In particular, Δ is a J_t-holomorphic curve for all t. By the positivity of intersections for J_t-holomorphic curves, each holomorphic curve in the foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ intersects Δ transversally in a single point.

We define a diffeomorphism $f : \Delta \to \Delta$ by $f(x) = y$, where $y \in \Delta$ is the unique point such that the J-holomorphic curve in $\mathcal{F}_1$ through y intersects the J-holomorphic curve in $\mathcal{F}_0$ through x on L. Then $f(x) \neq x$ for all $x \in \Delta$.

As in the previous section, for a point $x \in \Delta$ we denote its image under the antipodal map by $-x$. Then the J_0-holomorphic curve in $\mathcal{F}_0$ through x intersects the J_0-holomorphic curve in $\mathcal{F}_1$ through $-x$ on $\overline{\Delta}$ for all $x \in \Delta$.

We can apply the theorem of section 2 to get the following.

Lemma 7 *There exists an isotopy $g_t : \Delta \to \Delta$, $0 \leq t \leq 1$, with $g_0 = \mathrm{id}$, $g_1 = -f^{-1}$ and $g_t(x) \neq -x$ for all t and $x \in \Delta$.*

We now define maps $\phi_t : S^2 \times S^2 \to S^2 \times S^2$ by requiring that ϕ_t maps the J_t-holomorphic curves in $\mathcal{F}_0$ and $\mathcal{F}_1$ to the corresponding J_0-holomorphic foliations, the J_t-holomorphic curve in $\mathcal{F}_0$ through $x \in \Delta$ maps to the J_0-holomorphic curve in $\mathcal{F}_0$ through x and the J_t-holomorphic curve in $\mathcal{F}_1$ through x maps to the J_0-holomorphic curve in $\mathcal{F}_1$ through $g_t(x)$.

Then $\phi_0 = \mathrm{id}$, $\phi_1(L) = \overline{\Delta}$ and $\phi_t(\Delta)$ is disjoint from $\overline{\Delta}$ for all t. Let $L_t = \phi_t^{-1}(\overline{\Delta})$, so L_t gives a smooth isotopy from L to $\overline{\Delta}$ in $S^2 \times S^2 \setminus \Delta$.

Also, $\phi_{t*}(J_t)$ is tamed by the split form ω, and we see from this that $\phi_t(\Delta)$ is a symplectic submanifold for all t.

For fixed t, set $\omega_s = s\phi_t^*(\omega) + (1-s)\omega$. This is a symplectic form for all $0 \leq s \leq 1$. It is clearly closed and is symplectic since it tames J_t. We note that Δ is symplectic for all ω_s and, if $t = 0$ or $t = 1$, L_t is Lagrangian with respect to all ω_s. Hence by an application of Moser's theorem we can find a diffeomorphism ψ_t of $S^2 \times S^2$ such that $\psi_t^*(\omega) = \phi_t^*(\omega)$. The ψ_t can be chosen to vary smoothly with t, to fix Δ and such that $\psi_0 = \mathrm{id}$ and ψ_1 fixes L. To see this, we recall that Moser's method involves writing $\omega_s = \omega_0 + d\alpha_s$ and studying the flow of the vectorfield X_s defined by $X_s \lrcorner \omega_s = \frac{d\alpha_s}{ds}$. The definition implies that $\mathcal{L}_{X_s}\omega_s = d(\frac{d\alpha_s}{ds}) = \frac{d\omega_s}{ds}$. We have the freedom in this construction to add any smooth family of exact 1-forms β_s to the α_s. These β_s can be chosen such that $\alpha_s + \beta_s$ vanishes on the symplectic normal bundle to Δ and, if $t = 0$ or $t = 1$, on the tangent bundle to L_t. Then the flow fixes Δ and, if $t = 0$ or $t = 1$, also fixes L_t.

Thus $\psi_t(L_t)$ is a Lagrangian isotopy from L to $\overline{\Delta}$ inside $S^2 \times S^2 \setminus \Delta$ as required.

4   Manifolds with 1-handles

We now consider the class of convex symplectic manifolds constructed by adding 1-handles to the unit cotangent bundle $T^1 S^2$. Our first observation is that any such manifold M can be symplectically embedded in $(S^2 \times S^2, \omega)$, after perhaps scaling the symplectic form. This follows from the methods of [2]. We can arrange that the zero-section in $T^1 S^2$ again becomes identified with $\overline{\Delta}$ and the boundary of M is a smooth hypersurface Σ of contact-type in $S^2 \times S^2$.

We plan to find families of almost-complex structures J_t on $S^2 \times S^2$ and diffeomorphisms $f_t : \Delta \to \Delta$ such that the J_t-holomorphic curves in $\mathcal{F}_0$ through points x and the J_t-holomorphic curves in $\mathcal{F}_1$ through $f_t(x)$ intersect on embed-

ded spheres $L_t \subset M$ with $L_0 = \overline{\Delta}$ and $L_1 = L$. The almost-complex structures can be constructed by deforming J_0 in a neighbourhood of Σ and, for t close to 0 or 1, also in a neighbourhood of $\overline{\Delta}$ or L.

Suppose that we perform the operation of stretching-the-neck along Σ. That is, we symplectically identify a neighbourhood of Σ in $S^2 \times S^2$ with $((-\epsilon, \epsilon) \times \Sigma, d(e^t \alpha))$, where α is a fixed contact form on Σ. We can then produce a manifold A_N by replacing this neighbourhood by $(-N, N) \times \Sigma$. Our original almost-complex structure can be extended over $(-N, N) \times \Sigma$ to be translation invariant and the symplectic form can be extended over $(-N, N) \times \Sigma$ such that A_N is symplectomorphic to $(S^2 \times S^2, \omega)$ via a symplectomorphism equal to the identity outside $(-N, N) \times \Sigma$. Under this symplectomorphism we can think of stretching the neck as studying a family of almost-complex structures J_N on $S^2 \times S^2$ which degenerate along Σ as $N \to \infty$.

At the same time, we can deform the almost-complex structure along the boundary of tubular neighborhoods U_0 or U_1 of $L_0 = \overline{\Delta}$ or $L_1 = L$ respectively. Stretching to length N on both contact hypersurfaces Σ and ∂U_i we obtain almost-complex structures $J_{N,0}$ and $J_{N,1}$. There exist smooth families of almost-complex structures $J_{N,t}$ connecting $J_{N,0}$ and $J_{N,1}$ which are fixed on the tubular neighborhoods of Σ and in the complement of M.

Following the work of Hofer, see [6], and as in [4], after taking suitable subsequences, for $i, j = 0, 1$ families of $J_{N,i}$-holomorphic curves in $\mathcal{F}_j$ will converge to unions of finite energy planes as $N \to \infty$. The limiting finite energy planes can be chosen to foliate three symplectic manifolds with cylindrical ends, namely the completion W of the complement of M in $S^2 \times S^2$ with an end symplecto-morphic to the negative symplectization of Σ, that is $((-\infty, 0) \times \Sigma, d(e^t \alpha))$, the completion of U_i, which will be a copy of $T^* S^2$, and the completion of $M \setminus U_i$ with two ends symplectomorphic to the positive symplectization of Σ and the negative symplectization of the boundary of U_i.

The foliations of the completion of U_i were determined in [4]. For U_i and

its almost-complex structure suitably chosen, the Reeb flow on ∂U_i is foliated by closed orbits, and exactly one curve in each foliation is asymptotic to each closed orbit. Also, each curve in the foliation from $\mathcal{F}_0$ intersects in a single point each curve in the foliation from $\mathcal{F}_1$ provided that the curves have different asymptotic limits. By the positivity of intersections, any intersections of limiting finite energy planes must also be seen as intersections of holomorphic spheres in the foliations $\mathcal{F}_0$ and $\mathcal{F}_1$. It follows that finite energy curves in the foliations of W and the completion of $M \setminus U_i$ either have the same image or are disjoint. For, if any of these curves were to intersect in an isolated point we could find $J_{N,i}$-holomorphic curves in $\mathcal{F}_0$ and $\mathcal{F}_1$, for N sufficiently large, with intersection number at least two. There would be one intersection point near our point in W or $M \setminus U_i$ and another inside U_i. This gives a contradiction. Thus in particular the two foliations of W must coincide. Two curves in different homotopy classes which are disjoint in the completion of M come from taking a limit of $J_{N,i}$-holomorphic curves through the same point of Δ. The resulting finite energy curves in W coincide.

We can also obtain finite energy foliations of W and the completion of M by stretching the neck only along Σ and studying limits of $J_{N,t}$-holomorphic curves. A further limit of the $J_{N,0}$ and $J_{N,1}$ finite energy foliations of M as we degenerate the almost-complex structure along ∂U_i for $i = 0, 1$ would give our foliations of completions of $M \setminus U_i$ and U_i as above. If we take a diagonal subsequence of $N \to \infty$ then for a countable dense subset of $t \in [0, 1]$ the corresponding foliations of W and M can be assumed to arise from the same subsequence of $N \to \infty$. Since the $J_{N,t}$ are all equal outside of M, for each t we obtain a finite energy foliation of W with respect to the same almost-complex structure. For any fixed N, as $t \to t_0$, the $J_{N,t}$-holomorphic foliations of $S^2 \times S^2$ converge to the J_{N,t_0}-holomorphic foliations. Taking a further limit, we see that for t in our dense subset, and hence for all t since the subset was arbitrary, the finite energy foliations of W vary continuously with t. Suppose that the contact

form α on Σ and the almost-complex structure on W are chosen generically so that periodic orbits of the Reeb flow on Σ are isolated and embedded finite energy curves in W appear in families whose dimension is as predicted by the index theorem, see [1] and [5]. Then the following is true.

Lemma 8 *The deformation index of generic finite energy curves C in the foliation of W satisfies* index(C) ≤ 2.

Since 2-dimensional families of curves are needed to foliate W, this lemma implies that all finite energy planes sufficiently close to a curve C in our foliation actually appear in the foliation. In particular, as t varies the corresponding two finite energy foliations of W remain the same. We will use this result to draw our conclusion about the Lagrangian isotopy classes and then prove Lemma 8 at the end of the section.

Another result coming from the analysis in [4] is that each curve in the foliations of the completion of U_i coming from $\mathcal{F}_0$ and $\mathcal{F}_1$ intersects L_i transversally in a single point. Hence, for N sufficiently large the $J_{N,i}$ holomorphic curves in $\mathcal{F}_0$ and $\mathcal{F}_1$ must also intersect L_i transversally.

We now study the foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ corresponding to the family of almost-complex structures $J_{N,t}$ on $S^2 \times S^2$ for large N. As in section 3 we can find corresponding diffeomorphisms $f_{N,t} : \Delta \to \Delta$ such that $\overline{\Delta} = L_0$ consists of the intersections of $J_{N,0}$-holomorphic spheres in $\mathcal{F}_0$ through points $x \in \Delta$ with the spheres in $\mathcal{F}_1$ through $f_{N,0}(x)$, and $L = L_1$ consists of the intersections of $J_{N,1}$-holomorphic curves in $\mathcal{F}_0$ through points $x \in \Delta$ with the spheres in $\mathcal{F}_1$ through $f_{N,1}(x)$. By the result in section 2 we may assume that $f_{N,t}(x) \neq x$ for all x, t. As $N \to \infty$, the convergence of $J_{N,0}$ and $J_{N,1}$-holomorphic spheres implies that we may assume that the $f_{N,0}$ and $f_{N,1}$ converge to continuous maps $f_{\infty,0}$ and $f_{\infty,1}$ and hence that the $f_{N,t}$ converge to maps $f_{\infty,t}$ satisfying $f_{\infty,t}(x) \neq x$ for all t and x.

Next we define a family of embedded spheres by letting L_t be the union of the intersections of $J_{N,t}$-holomorphic curves in $\mathcal{F}_0$ through points x with $J_{N,t}$-

holomorphic curves in $\mathcal{F}_1$ through $f_{N,t}(x)$. We claim that for N sufficiently large $L_t \subset M$ for all t. For otherwise there exists a subsequence $N \to \infty$ such that for each N there exists a t and $x \in \Delta$ with the $J_{N,t}$ holomorphic sphere in $\mathcal{F}_0$ through x intersecting the $J_{N,t}$ holomorphic sphere in $\mathcal{F}_1$ through $f_{N,t}(x) \neq x$ in a point outside of M. Taking a limit as $N \to \infty$ we then have that the two foliations of W corresponding to limits of $J_{N,t}$-holomorphic spheres in $\mathcal{F}_0$ and $\mathcal{F}_1$ will not coincide, a contradiction since all of these foliations are identical for all t.

Following the method of section 3, we can find a family of symplectic forms ω_t on $S^2 \times S^2$ such that L_t is Lagrangian with respect to ω_t. The ω_t restrict to exact symplectic forms on M, say $\omega_t = d\alpha_t$ which are tamed by J_t. In a tubular neighbourhood $V = (-\epsilon, 0) \times \Sigma$ of the boundary $\Sigma = \{0\} \times \Sigma$ of M, define a function $\chi : V \to [0, 1)$ such that $\chi(r, y)$ is an increasing function of r, $\chi(r, y) = 0$ for r close to $-\epsilon$ and $\chi(r, y) = 1$ for r close to 0. Then, first scaling α_t if necessary, we can replace it by $\beta_t = (1 - \chi)\alpha_t + \chi e^r \alpha$ in V. The new form $\omega_t = d\beta_t$ will still be symplectic and tamed by J_t (for α_t suitably scaled) but now agrees with ω near Σ. Assuming V to be disjoint from all L_t, the submanifolds L_t will still by Lagrangian with respect to ω_t.

We now apply Moser's method as in section 3 to find a symplectomorphism between (M, ω_t) and (M, ω) and thereby isotope the L_t into Lagrangian submanifolds of (M, ω). As before, this can be arranged to fix L_0 and L_1 and now also the neighbourhood V. Thus it gives our Lagrangian isotopy as required.

Proof of Lemma 8

Let C be a finite energy curve in the foliation of W. The curve C will be one component of a limit of holomorphic spheres. The other components can be assumed to curves D_i in the symplectization of Σ and curves E_j in the completion of M. Since the finite energy curves are limits of curves of genus 0 and as in [4] the limiting curve has only one component in W, the D_i and E_j have only one positive asymptotic limit. We recall that the index

formula for a finite energy curve F depends upon a trivialization of the contact planes along its asymptotic limits (which are Reeb orbits in certain contact manifolds). We can then define a Conley-Zehnder index for each asymptotic limit and a Chern class $c_1(F)$ relative to these trivializations. Suppose that the positive asymptotic limits have Conley-Zehnder indices μ_k^+ for $1 \leq k \leq m$ and the negative asymptotic limits have index μ_l^- for $1 \leq l \leq n$. If the asymptotic limits are nondegenerate the formula for the deformation index of F modulo reparameterizations is

$$\text{index}(F) = -(2 - m - n) + 2c_1(F) + \sum_{k=1}^{m} \mu_k^+ - \sum_{l=1}^{n} \mu_l^-.$$

In our case, a global trivialization of the contact planes in $T^1 S^2$ extends over any 1-handles to a trivialization of $\xi = \{\alpha = 0\}$ on Σ and we compute our indices relative to this. Then the curves in the symplectization of Σ and the completion of M have Chern class 0 and our curve C has $c_1(C) = 2$.

A curve E_j has a single positive asymptotic limit. If this has index μ_j^+ then we obtain

$$\text{index}(E_j) = -1 + \mu_j^+.$$

But for a generic choice of almost-complex structure this index must be non-negative and so $\mu_j^+ \geq 1$ for all j.

Suppose that a curve D_i has a positive asymptotic limit with index μ_i^+ and m_i negative asymptotic limits with index μ_{ik}^- for $1 \leq k \leq m_i$. Then the index formula becomes

$$\text{index}(D_i) = -1 + m_i + \mu_i^+ - \sum_{k=1}^{m_i} \mu_{ik}^-.$$

Again generically this index must be nonnegative. Therefore if all of the negative asymptotic limits are positive asymptotic limits of curves E_j we obtain $\mu_i^+ \geq 1$. By an induction on the number of levels of the limiting finite energy curve we deduce that in fact $\mu_i^+ \geq 1$ for all curves D_i.

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Finally we look at C. It has only negative asymptotic limits and if these have index μ_l^- for $1 \leq l \leq n$ then

$$\text{index(C)} = 2 + \text{n} - \sum_{l=1}^{n} \mu_l^-.$$

But all of these negative asymptotic limits are positive limits of curves D_i or E_j. Hence $\mu_l^- \geq 1$ for all l and so index(C) ≤ 2 as required.

References

[1] Y. Eliashberg, A. Givental and H. Hofer *Introduction to symplectic field theory*, GAFA 2000 (Tel Aviv, 1999). Geom. Funct. Anal. 2000, Special Volume, Part II, 560–673.

[2] Y. Eliashberg and M. Gromov, *Convex Symplectic Manifolds*, Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989), 135-162, Proc. Sympos. Pure Math., 52, Part 2.

[3] M. Gromov, Pseudo-holomorphic curves in symplectic manifolds, *Inv. Math.*, 82(1985), 307-347.

[4] R. Hind, Lagrangian spheres in $S^2 \times S^2$, preprint.

[5] H. Hofer, K. Wysocki and E. Zehnder, Properties of pseudoholomorphic curves in symplectisations III: Fredholm theory, *Topics in nonlinear analysis*, 381-475, Prog. Nonlinear Differential Equations Appl., 35, Birkhäuser, Basel, 1999.

[6] H. Hofer, K. Wysocki and E. Zehnder, Finite energy foliations of tight three-spheres and Hamiltonian dynamics, *Ann. of Math.*, 157(2003), 125-255.

[7] D. McDuff, The local behaviour of J-holomorphic curves in almost complex 4-manifolds, *J. Diff. Geom.*, 34(1991), 143-164.

[8] P. Seidel, Lagrangian two-spheres can be symplectically knotted, *J. Diff. Geom.*, 52(1999), 145-171.

[9] B. Siebert and G. Tian, On the holomorphicity of genus two Lefschetz fibrations, preprint math.SG/0305343.

[10] M. Symington, Symplectic rational blowdowns, *J. Diff. Geom.*, 50(1998), 505-518.

[11] W.P. Thurston, *Three-dimensional geometry and topology, Vol. I*, edited by S. Levy, Princeton University Press, 1997.